10.6 ~ Combinations and Probability

Daily Objectives

1. Learn the notation and formula for the number of combinations.
2. Understand the relationships between permutations and combinations.
3. Use combination numbers to calculate probabilities.

Example 1: At the first meeting of the International Club, the members get acquainted by introducing themselves and shaking hands. Each member shakes hands with every other member exactly once. How many handshakes are there in each of the situations below?

a. Three people meet.

AB
AC
BC

3

$\frac{3 \cdot 2}{2} = 3$

b. Four people meet.

AB BC CD
AC BD
AD

6

$\frac{4 \cdot 3}{2}$

c. Five people meet.

AB BC CD DE
AC BD CE
AD BE
AE

10

$\frac{5 \cdot 4}{2}$

d. Fifteen people meet.

$14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$

Same as ${}_{15}P_2 = 15 \cdot 14$ but there are half as many because AB and BA can only be counted once, so...

$\frac{15 \cdot 14}{2} = 105$

What is the difference between a permutation and a combination?

Permutations order matters so AB and BA are both counted.
Combinations order does not matter so AB and BA only count as one.

Example 2: Anna, Ben, Chang, and Dena are members of the International Club, and they have volunteered to be on a committee that will arrange a reception for exchange students. Usually there are only three students on the committee. How many different three-member committees could be formed with these four students?

6 permutations of each

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

$$\frac{4!}{3!\,1!} = \frac{4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 4$$

* Each of the 4 committees in the top row can represent all of the 3! or 6 permutations listed in its column. $_4C_3$ is $\frac{1}{6}$ the number of permutations.

Combinations

A **combination** is a grouping of some or all of the objects from a set without regard to order.

The number of combinations of n objects chosen r at a time $(r \leq n)$ is

$$_nC_r = \frac{_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

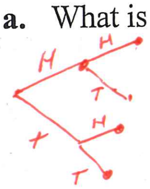
How does the formula for *combinations* relate to the one for *permutations*? Why is this so?

$$_4P_3 = \frac{4\cdot 3\cdot 2}{3!}$$

Take the number of permutations and divide it by the number of permutations of the number of objects chosen.

Example 3: Suppose a coin is flipped 10 times.

a. What is the probability that it will land <u>heads</u> **exactly** five times?



COUNTING PRINCIPLE

FLIP: 1 2 3 4 5 6 7 8 9 10

$2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 = 2^{10} = 1024$ possibilities

WAYS TO GET 5 HEADS $= {}_{10}C_5$ (order doesn't matter)

$$\frac{10!}{5!\,5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{5\cdot 4\cdot 3\cdot 2\cdot 1\;\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 252$$

$$\frac{252}{1024} = \boxed{.246}$$

b. What is the probability that it will land **exactly** five times, including on the third flip?

Heads on third $\frac{1}{2}$

$_9C_4$ on remaining

$$\frac{_9C_4}{2^9} = \frac{126}{512}$$

or $\frac{_9C_4}{2^{10}}$

$$\frac{1}{2}\cdot\frac{126}{512} = \frac{126}{1024} = \boxed{.123}$$

Investigation – Winning the Lottery

Consider a state lottery called Lotto 47. Twice a week, players select six different numbers between 1 and 47. The state lottery commission also selects six numbers between 1 and 47. Selection order doesn't matter, but a player needs to match all six numbers to win Lotto 47.

Step 1: Follow these directions with your class to simulate playing Lotto 47.

For five minutes, write down as many sets of six different numbers as you can. Write *only* integers between 1 and 47.

After 5 minutes of writing, everyone stands up.

Your teaching will generate a random integer 1-47. Cross out all of your sets of six numbers that do not contain the given number. If you cross out all of your sets, sit down.

Your teacher will generate a second number, third number, etc… Follow the previous rule for each new integer. Your teacher will continue generating different random numbers until no one is standing or six numbers have been generated..

a. What is the probability that any one set of six numbers wins?

$$\frac{1}{{}_{47}C_6} = \frac{1}{\frac{47\cdot 46\cdot 45\cdot 44\cdot 43\cdot 42}{6\cdot 5\cdot 4\cdot 3\cdot 2}} = \frac{1}{10{,}737{,}573} \approx .0000000931$$

b. At $1 for each set of six numbers, how much did each of you spend in your five minutes?

c. How much did the entire class spend?

d. Estimate the probability that someone in your class wins. Explain how you determined this estimate.

$\frac{c}{10,737,573} =$

e. Estimate the probability that someone in your school would win if everyone in the school participated in this activity. Explain how you determined this estimate.

$1300 \times \frac{c}{\text{\# of students}}$

f. If each possible set of sit numbers were written on a 1-inch chip and if all the chips were laid end to end, how long would the line of chips be

a. In feet?

$$\frac{10{,}737{,}573 \text{ inches}}{} \left| \frac{1 \text{ foot}}{12 \text{ in}} \right.$$

$\boxed{894{,}797.75 \text{ ft}}$

b. In miles?

$$\frac{10{,}737{,}573 \text{ in}}{} \left| \frac{1 \text{ ft}}{12 \text{ in}} \right| \frac{1 \text{ mi}}{5280 \text{ ft}}$$

$\boxed{169.5 \text{ mi}}$